

CARTAN COHERENT CONFIGURATIONS

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ABSTRACT. The Cartan scheme \mathcal{X} of a finite group G with a (B, N) -pair is defined to be the coherent configuration associated with the action of G on the right cosets of the Cartan subgroup $B \cap N$ by the right multiplications. It is proved that if G is a simple group of Lie type, then asymptotically, the coherent configuration \mathcal{X} is 2-separable, i.e., the array of 2-dimensional intersection numbers determines \mathcal{X} up to isomorphism. It is also proved that in this case, the base number of \mathcal{X} equals 2. This enables us to construct a polynomial-time algorithm for recognizing the Cartan schemes when the rank of G and order of the underlying field are sufficiently large. One of the key points in the proof of the main results is a new sufficient condition for an arbitrary homogeneous coherent configuration to be 2-separable.

1. INTRODUCTION

A well-known general problem in algebraic combinatorics is to characterize an association scheme \mathcal{X} up to isomorphism by a certain set of parameters [3]. A lot of such characterizations are known when \mathcal{X} is the association scheme of a classical distance regular graph [4]. In most cases, the parameters can be chosen as a part of the intersection number array of \mathcal{X} . However in general, even the whole array does not determine the scheme \mathcal{X} up to isomorphism. Therefore, it makes sense to consider the m -dimensional intersection numbers, $m \geq 1$, introduced in [11] for an arbitrary coherent configuration (for $m = 1$, these numbers are ordinary intersection numbers; the exact definitions can be found in Section 2). It was proved in [11] that every Johnson, Hamming or Grassmann scheme is 2-separable, i.e., is determined up to isomorphism by the array of 2-dimensional intersection numbers.

In a recent paper [1], a generalized notion of distance regularity in buildings was introduced. According to [24], there is a natural 1-1 correspondence between the class of all buildings and the class of special homogeneous coherent configurations called the Coxeter schemes (see also [23, Chapter 12]). In this language, the Tits theorem on spherical buildings says that if \mathcal{X} is a finite Coxeter scheme of rank at least 3, then there exists a group G acting on a set Ω such that

$$(1) \quad \mathcal{X} = \text{Inv}(G, \Omega)$$

where $\text{Inv}(G, \Omega)$ is the coherent configuration of G , i.e., the pair (Ω, S) with $S = \text{Orb}(G, \Omega \times \Omega)$. Moreover, in this case, G is a group with a (B, N) -pair. Thus, a characterization of the coherent configuration (1) with such G by the m -dim intersection numbers with small m could be consider as a generalization of the above mentioned results on the association schemes of classical distance regular graphs to the noncommutative case.

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In the present paper, we are interested in coherent configurations (1) in the case when G is a finite group with a (B, N) -pair and \mathcal{X} is a *Cartan scheme* of G in the following sense.

Definition 1.1. *The Cartan scheme of G with respect to (B, N) is defined to be the coherent configuration (1), where $\Omega = G/H$ consists of the right cosets of the Cartan subgroup $H = B \cap N$ and G acts on Ω by right multiplications.*

Note that the permutation group induced by the action of G is transitive and the stabilizer of the point $\{H\}$ coincides with H . In a Coxeter scheme of rank at least 3, the point stabilizer equals B . Therefore, every Coxeter scheme is a quotient of a suitable Cartan scheme.

The separability problem [12] consists in finding the smallest m , for which a coherent configuration is m -separable. The separability problem (in particular, for a Cartan scheme) is easy to solve if the group H is trivial. Indeed, in this case, the permutation group induced by G is regular and the corresponding coherent configuration is 1-separable. The following theorem gives a partial solution to the separability problem for Cartan schemes when G is a finite simple group of Lie type, and hence with a (B, N) -pair. In what follows, we denote by \mathcal{L} the class of all simple groups of Lie type including all exceptional groups and all classical groups $\Phi(l, q)$, for which $l \geq l_0$ and $q \geq al$, where the values of l_0 and a are given in the last two columns of Table 2 at page 17.

Theorem 1.2. *The Cartan scheme \mathcal{X} of every finite simple group $G \in \mathcal{L}$ is 2-separable.*

As a byproduct of the proof of Theorem 1.2, we are able to estimate the base number of a Cartan scheme satisfying the hypothesis of this theorem (as to the exact definition, we refer to Subsection 2.2; see also [12] and [2, Sec. 5], where the base number was called the EP-dimension). The base number $b(\mathcal{X})$ of a coherent configuration \mathcal{X} can be thought as a combinatorial analog of the base number of a permutation group, which is the minimal number of points such that only identity of the group leave each of them fixed. In fact, for the coherent configuration (1), we have

$$(2) \quad b(G) \leq b(\mathcal{X}),$$

where $b(G)$ is the base number of the permutation group induced by G . Moreover, in this case, obviously, $b(G) = 1$ if and only if $b(\mathcal{X}) = 1$. In general, $b(G)$ can be much smaller than $b(\mathcal{X})$. The following theorem shows that this does not happen for the Cartan schemes in question.

Theorem 1.3. *Let \mathcal{X} be the Cartan scheme of a group $G \in \mathcal{L}$. Then $b(\mathcal{X}) \leq 2$ and $b(\mathcal{X}) = 1$ if and only if the group H is trivial.*

Let us deduce Theorems 1.2 and 1.3 from the results, which proofs occupy the most part of the paper. Let \mathcal{X} be the Cartan scheme of a group $G \in \mathcal{L}$. Denote by c and k the indistinguishing number and the maximum valency of \mathcal{X} , respectively.¹ Translating these invariants into the group-theoretic language, we prove in

¹In the complete colored graph representing \mathcal{X} , k is the maximum number of the monochrome arcs incident to a vertex, and c is the maximum number of triangles with fixed base, the other two sides of which are monochrome arcs.

Theorem 4.1 that in our case

$$(3) \quad 2c(k-1) < n$$

where $n = |\Omega|$. The proof of this inequality forms the group-theoretic part of the whole proof. The combinatorial part of the proof is to analyze one point extension of a homogeneous coherent configuration, for which inequality (3) holds; here the point extension can be thought as a combinatorial analog of the point stabilizer of a permutation group. In this way, we prove Theorem 3.1, which implies that the one point extension of \mathcal{X} is 1-regular (see Subsection 2.2). It immediately follows that $b(\mathcal{X}) \leq 2$, which proves Theorem 1.3. Finally, in view of Theorem 3.1, the 2-separability of \mathcal{X} is a direct consequence of Theorem 2.5 obtained by a combination of two results in [12], so Theorem 1.2 holds.

When the rank of a simple group G of Lie type is small, inequality (3) does not generally hold, but the statements of Theorems 1.2 and 1.3 may still be true. For example, this happens in the following case.

Theorem 1.4. *Let \mathcal{X} be the Cartan scheme of the group $\text{PSL}(2, q)$, where $q > 3$. Then \mathcal{X} is 2-separable and $b(\mathcal{X}) = 2$.*

We believe that the Cartan scheme of every simple group of Lie type is 2-separable. Moreover, as in the case of classical distance regular graphs, it might be that in most cases such a scheme is 1-separable, i.e., is determined up to isomorphism by the intersection numbers. In this way, one could probably use more subtle results on the structure of finite simple groups and a combinatorial technique in spirit of [15].

From the computational point of view, Theorems 1.2 and 1.3 can be used for testing isomorphism and recognizing the Cartan schemes satisfying the hypothesis of Theorem 1.2. For this aim, it is convenient to represent a coherent configuration (Ω, S) as a complete colored graph with vertex set Ω , in which the color classes of arcs coincide with relations of the set S (the vertex colors match the colors of the loops). It is assumed that isomorphisms of such colored graphs preserve the colors.

Theorem 1.5. *Let \mathcal{G}_n (resp. \mathcal{K}_n) be the class of all colored graphs (resp. the colored graphs of Cartan schemes of the groups in \mathcal{L}) with n vertices. Then the following problems can be solved in polynomial time in n :*

- (1) *given $D \in \mathcal{G}_n$, test whether $D \in \mathcal{K}_n$, and (if so) find the corresponding groups G , B , and N ;*
- (2) *given $D \in \mathcal{K}_n$ and $D' \in \mathcal{G}_n$, find the set $\text{Iso}(D, D')$.*

To make the paper possibly self-contained, we cite the basics of coherent configurations in Section 2. Theorems 3.1 and 4.1, from which we have deduced Theorems 1.2 and 1.3, are proved in Section 3 and Sections 4,5, respectively. Finally, Theorems 1.4 and 1.5 are proved in Sections 6 and 7, respectively.

Notation. Throughout the paper, Ω denotes a finite set.

The diagonal of the Cartesian product $\Omega \times \Omega$ is denoted by 1_Ω ; for any $\alpha \in \Omega$ we set $1_\alpha = 1_{\{\alpha\}}$.

For a relation $r \subset \Omega \times \Omega$, we set $r^* = \{(\beta, \alpha) : (\alpha, \beta) \in r\}$ and $\alpha r = \{\beta \in \Omega : (\alpha, \beta) \in r\}$ for all $\alpha \in \Omega$.

For $S \in 2^{\Omega^2}$, we denote by S^\cup the set of all unions of the elements of S , and put $S^* = \{s^* : s \in S\}$ and $\alpha S = \cup_{s \in S} \alpha s$, where $\alpha \in \Omega$.

For $g \in \text{Sym}(\Omega)$, we set $\text{Fix}(g) = \{\alpha \in \Omega : \alpha^g = \alpha\}$; in particular, if χ is the permutation character of a group $G \leq \text{Sym}(\Omega)$, then $\chi(g) = |\text{Fix}(g)|$ for all $g \in G$.

The identity of a group G is denoted by e ; the set of non-identity elements in G is denoted by $G^\#$.

2. COHERENT CONFIGURATIONS

2.1. Main definitions. Let Ω be a finite set, and let S be a partition of $\Omega \times \Omega$. The pair $\mathcal{X} = (\Omega, S)$ is called a *coherent configuration* on Ω if $1_\Omega \in S^\cup$, $S^* = S$, and given $r, s, t \in S$, the number

$$c_{rs}^t = |\alpha r \cap \beta s^*|$$

does not depend on the choice of $(\alpha, \beta) \in t$. The elements of Ω , S , and S^\cup are called the *points*, *basis relations*, and *relations* of \mathcal{X} , respectively. The numbers $|\Omega|$, $|S|$, and c_{rs}^t are called the *degree*, *rank*, and *intersection numbers* of \mathcal{X} . The basis relation containing the pair $(\alpha, \beta) \in \Omega \times \Omega$ is denoted by $r(\alpha, \beta)$.

The point set Ω is a disjoint union of *fibers*, i.e., the sets $\Gamma \subseteq \Omega$, for which $1_\Gamma \in S$. For any basis relation $r \in S$, there exist uniquely determined fibers Γ and Δ such that $r \subseteq \Gamma \times \Delta$. Moreover, the number $|\gamma r| = c_{r\Gamma}^t$ with $t = 1_\Gamma$, does not depend on the choice of $\gamma \in \Gamma$. This number is called the *valency* of r and denoted n_r . The maximum of all valences is denoted by $k = k(\mathcal{X})$.

A point $\alpha \in \Omega$ of the coherent configuration \mathcal{X} is called *regular* if

$$|\alpha r| \leq 1 \quad \text{for all } r \in S.$$

One can see that the set of all regular points is the union of fibers. If this set is not empty, then the coherent configuration \mathcal{X} is said to be *1-regular*.

The coherent configuration \mathcal{X} is said to be *homogeneous* if $1_\Omega \in S$. In this case, $n_r = n_{r^*} = |\alpha r|$ for all $r \in S$ and $\alpha \in \Omega$. Moreover, the relations

$$(4) \quad c_{r^*s^*}^{t^*} = c_{sr}^t \quad \text{and} \quad n_t c_{rs}^{t^*} = n_r c_{st}^{r^*} = n_s c_{tr}^{s^*}$$

hold for all $r, s, t \in S$. We observe that in the homogeneous case, a coherent configuration is 1-regular if and only if it is a thin scheme in the sense of [23].

2.2. Point extensions and the base number. There is a natural partial order \leq on the set of all coherent configurations on the same set. Namely, given two coherent configurations $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega, S')$, we set

$$\mathcal{X} \leq \mathcal{X}' \Leftrightarrow S^\cup \subseteq (S')^\cup.$$

The minimal and maximal elements with respect to this ordering are the *trivial* and *complete* coherent configurations: the basis relations of the former one are the reflexive relation 1_Ω and (if $n > 1$) its complement in $\Omega \times \Omega$, whereas the basis relations of the latter one are singletons.

Given two coherent configurations \mathcal{X}_1 and \mathcal{X}_2 on Ω , there is a uniquely determined coherent configuration $\mathcal{X}_1 \cap \mathcal{X}_2$ also on Ω , the relation set of which is $(S_1)^\cup \cap (S_2)^\cup$, where S_i is the set of basis relations of \mathcal{X}_i , $i = 1, 2$. This enables us

to define the *point extension* $\mathcal{X}_{\alpha,\beta,\dots}$ of a coherent configuration $\mathcal{X} = (\Omega, S)$ with respect to the points $\alpha, \beta, \dots \in \Omega$ as follows:

$$\mathcal{X}_{\alpha,\beta,\dots} = \bigcap_{\mathcal{Y}: S \subseteq T^\cup, 1_\alpha, 1_\beta, \dots \in T^\cup} \mathcal{Y},$$

where \mathcal{Y} is the coherent configuration (Ω, T) . In other words, $\mathcal{X}_{\alpha,\beta,\dots}$ can be defined as the smallest coherent configuration on Ω that is larger than or equal to \mathcal{X} and has singletons $\{\alpha\}, \{\beta\}, \dots$ as fibers. This configuration can also be considered as the refinement of the color graph associated with \mathcal{X} , in which the points of α, β, \dots are colored in distinguished new colors. In particular, the extension can be efficiently constructed by the Weisfeiler-Leman algorithm (see Section 7).

Definition 2.1. A set $\{\alpha, \beta, \dots\} \subseteq \Omega$ is a *base* of the coherent configuration \mathcal{X} if the extension $\mathcal{X}_{\alpha,\beta,\dots}$ with respect to the points α, β, \dots is complete; the smallest cardinality of a base is called the *base number* of \mathcal{X} and denoted by $b(\mathcal{X})$.

It is easily seen that $0 \leq b(\mathcal{X}) \leq n - 1$, where $n = |\Omega|$, and the equalities are attained for the complete and trivial coherent configurations on Ω , respectively. It is also obvious that $b(\mathcal{X}) \leq 1$, whenever the coherent configuration \mathcal{X} is 1-regular.

2.3. Coherent configurations and permutation groups. Two coherent configurations $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ are called *isomorphic* if there exists a bijection $f : \Omega \rightarrow \Omega'$ such that the relation $s^f = \{(\alpha^f, \beta^f) : (\alpha, \beta) \in s\}$ belongs to S' for all $s \in S$. The bijection f is called an *isomorphism* from \mathcal{X} onto \mathcal{X}' ; the set of all of them is denoted by $\text{Iso}(\mathcal{X}, \mathcal{X}')$. The group $\text{Iso}(\mathcal{X}, \mathcal{X})$ contains a normal subgroup

$$\text{Aut}(\mathcal{X}) = \{f \in \text{Sym}(\Omega) : s^f = s, s \in S\}$$

called the *automorphism group* of \mathcal{X} .

Let $G \leq \text{Sym}(\Omega)$ be a permutation group, and let S be the set of orbits of the coordinatewise action of G on $\Omega \times \Omega$. Then

$$\text{Inv}(G) = \text{Inv}(G, \Omega) = (\Omega, S)$$

is a coherent configuration called the *coherent configuration of G* . It is homogeneous if and only if the group G is transitive. From [12, Corollary 3.4], it follows that a coherent configuration \mathcal{X} is 1-regular if and only if $\mathcal{X} = \text{Inv}(G)$, where G is a permutation group having a faithful regular orbit.

Let $G \leq \text{Sym}(\Omega)$ be a transitive group, $H = G_\alpha$ the stabilizer of a point α in G , and $\mathcal{X} = \text{Inv}(G)$ the coherent configuration of G . Then given a basis relation $s \in S$, one can form the set

$$(5) \quad D_s = \{g \in G : (\alpha, \alpha^g) \in s\},$$

which is, in fact, a double H -coset. It is well known that the mapping $s \mapsto D_s$ is a bijection from the set S of basis relations of \mathcal{X} onto the set of double H -cosets in G . Moreover, the intersection number c_{rs}^t is equal to the multiplicity, with which an element of D_t enters the product $D_r D_s$, divided by $|H|$. It follows that

$$(6) \quad n_s = \frac{|D_s|}{|H|} = \frac{|H|}{|H \cap H^g|}$$

for all $s \in S$ and $g \in D_s$ (the second equality follows from the first one, because $|D_s| = |HgH| = |g^{-1}HgH| = |H^gH|$). In particular, $k = k(\mathcal{X})$ is the ratio between the order of H and the minimal size of the intersection of H with its conjugate.

Lemma 2.2. *Let G be a transitive permutation group and $\mathcal{X} = \text{Inv}(G)$. If $b(\mathcal{X}) \leq 2$, then $G = \text{Aut}(\mathcal{X})$.*

Proof. Inequality (2) yields $b(G) \leq b(\mathcal{X}) \leq 2$. It follows that $H \cap H^g = 1$ for some $g \in G$, where $H = G_\alpha$. If s is the basis relation of \mathcal{X} with $D_s = HgH$, then (6) implies that αs is a faithful regular orbit of H . Hence

$$|G| = nk,$$

where n is the cardinality of underlying set of G . Since $\mathcal{X} = \text{Inv}(G) = \text{Inv}(\text{Aut}(\mathcal{X}))$, the above equality holds also for G replaced by $\text{Aut}(\mathcal{X})$. Thus,

$$|\text{Aut}(\mathcal{X})| = nk = |G|,$$

and we are done, because $G \leq \text{Aut}(\mathcal{X})$. \blacksquare

2.4. Indistinguishing number. Following [15], the sum of all intersection numbers $c_{ss^*}^r$ with fixed r is called the *indistinguishing number* of $r \in S$ and denoted by $c(r)$. It is easily seen that for all pairs $(\alpha, \beta) \in r$, we have

$$(7) \quad c(r) = |\Omega_{\alpha, \beta}|, \text{ where } \Omega_{\alpha, \beta} = \{\gamma \in \Omega : r(\gamma, \alpha) = r(\gamma, \beta)\}.$$

The maximum of the numbers $c(r)$, $r \neq 1_\Omega$, is called the *indistinguishing number* of the coherent configuration \mathcal{X} and denoted by $c(\mathcal{X})$.

The following lemma gives a formula for the indistinguishing number of the coherent configuration of a transitive permutation group. Recall that the fixity $\text{fix}(G)$ of a permutation group G is the maximum number of elements fixed by non-identity permutations [18].

Lemma 2.3. *Let $G \leq \text{Sym}(\Omega)$ be a transitive group, H a point stabilizer of G , and $\mathcal{X} = \text{Inv}(G)$. Then*

$$(8) \quad c(\mathcal{X}) = \max_{x \in G \setminus H} \left| \bigcup_{h \in H} \text{Fix}(hx) \right|.$$

In particular,

$$(9) \quad c(\mathcal{X}) \leq \max_{x \in G \setminus H} \sum_{h \in H} \chi(hx) \leq \text{fix}(G) \cdot |H|.$$

Proof. Let $r \in S$ and $(\alpha, \beta) \in r$. Then a point γ belongs to the set $\Omega_{\alpha, \beta}$ defined in (7) if and only if the pairs (γ, α) and (γ, β) belong to the same orbit of the group G acting on $\Omega \times \Omega$, and the latter happens if and only if γ is a fixed point of a permutation $x \in G$ moving α to β . Assuming without loss of generality that $H = G_\alpha$, we conclude that the set of all such x forms an H -coset C . Therefore,

$$(10) \quad c(r) = |\Omega_{\alpha, \beta}| = \left| \bigcup_{h \in H} \text{Fix}(hx) \right|$$

for any $x \in C$. Moreover, if $r \neq 1_\Omega$, then $C \neq H$. This proves equality (8). Furthermore, $|\text{Fix}(x)| = \chi(x) \leq \text{fix}(G)$ for any non-identity element $x \in G$. This implies that

$$\left| \bigcup_{h \in H} \text{Fix}(hx) \right| \leq \sum_{h \in H} \chi(hx) \leq \text{fix}(G) \cdot |H|.$$

Thus the second statement of the lemma follows from the first one. \blacksquare

We complete this subsection by a statement that helps to count the values of the permutation character of a transitive group.

Lemma 2.4. *Let $G \leq \text{Sym}(\Omega)$ be a transitive group, $\alpha \in \Omega$, and $H = G_\alpha$ the point stabilizer of α in G . Then for every $x \in G$*

$$(11) \quad \text{Fix}(x) \neq \emptyset \iff x^G \cap H \neq \emptyset.$$

Suppose, additionally, that there is a subgroup N with $H \leq N \leq N_G(H)$ such that every two elements of H conjugated in G are conjugated in N . If $x = h_0^{g_0}$, where $h_0 \in H$ and $g_0 \in G$, then

$$(12) \quad \text{Fix}(x) = \{\alpha^g \mid g \in NCg_0\},$$

where $C = C_G(h_0)$. Furthermore,

$$(13) \quad \chi(x) = \frac{|N : (C \cap N)| |C|}{|H|} = \frac{|N : (C \cap N)| |\Omega|}{|x^G|}.$$

Proof. Clearly, $\alpha^g \in \text{Fix}(x)$ if and only if $Hgx = Hg$, which holds if and only if there is $h \in H$ satisfying $x = h^g$. In particular, this yields (11).

To prove that the left-hand side of (12) is contained in the right-hand side, let $x = h_0^{g_0}$, that is the set $\text{Fix}(x)$ is nonempty. Suppose that g is an arbitrary element of G with $\alpha^g \in \text{Fix}(x)$. Then there is $h \in H$ such that $h^g = x = h_0^{g_0}$. Put $y = gg_0^{-1}$. Since the elements h_0 and $h = h_0^{y^{-1}}$ are conjugated in G , they are conjugated in N , so there is $n \in N$ with $h_0^{y^{-1}} = h_0^{n^{-1}}$. It follows that $y = nc$, where $c \in C$. Therefore, $g = ncg_0$, so $\alpha^g \in \text{Fix}(x)$ implies that $g \in NCg_0$. To establish the converse inclusion, for every $n \in N$, set $h = h_0^{n^{-1}}$. Then $h^{ncg_0} = h_0^{c g_0} = x$ for every $c \in C$. By the argument of the first paragraph, this proves $\alpha^{NCg_0} \subseteq \text{Fix}(x)$.

Obviously, $|NCg_0| = |N : (C \cap N)| |C|$. Now, the first equality in (13) is the direct consequence of (12), because $\alpha^g = \alpha^{g'}$ if and only if $g'g^{-1} \in H$. Since $|C| = |G|/|x^G|$ and $|G|/|H| = |\Omega|$, the second equality follows. ■

2.5. Algebraic isomorphisms and m -dimensional intersection numbers.

Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be coherent configurations. A bijection $\varphi : S \rightarrow S'$, $r \mapsto r'$ is called an *algebraic isomorphism* from \mathcal{X} to \mathcal{X}' if

$$(14) \quad c_{rs}^t = c_{r's'}^{t'}, \quad r, s, t \in S.$$

In this case, we say that \mathcal{X} and \mathcal{X}' are *algebraically isomorphic*. Each isomorphism f from \mathcal{X} to \mathcal{X}' naturally induces an algebraic isomorphism between these coherent configurations. The set of all isomorphisms inducing the algebraic isomorphism φ is denoted by $\text{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$. In particular,

$$\text{Iso}(\mathcal{X}, \mathcal{X}, \text{id}_S) = \text{Aut}(\mathcal{X})$$

where id_S is the identity on S . A coherent configuration \mathcal{X} is called *separable* if, for any algebraic isomorphism $\varphi : \mathcal{X} \rightarrow \mathcal{X}'$, the set $\text{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ is nonempty.

Saying that coherent configurations \mathcal{X} and \mathcal{X}' have the same intersection numbers, we mean that formula (14) holds for a certain algebraic isomorphism. Thus, the exact meaning of the phrase "the coherent configuration \mathcal{X} is determined up to isomorphism by the intersection numbers" consists in the fact that \mathcal{X} is separable.

Let $m \geq 1$ be an integer. According to [12], the m -extension of a coherent configuration \mathcal{X} with point set Ω is defined to be the smallest coherent configuration on Ω^m , which contains the Cartesian m -power of \mathcal{X} and for which the set $\text{Diag}(\Omega^m)$ is a union of fibers. The intersection numbers of the m -extension are called the

m -dimensional numbers of the configuration \mathcal{X} . Now, m -separable coherent configurations for $m > 1$ are defined essentially in the same way as for $m = 1$. The exact definition can be found in survey [12], whereas in the present paper, we need only the following result, which immediately follows from [12, Theorems 3.3 and 5.10].

Theorem 2.5. *Let \mathcal{X} be a coherent configuration admitting a 1-regular extension with respect to $m - 1$ points, $m \geq 1$. Then \mathcal{X} is m -separable.* ■

3. A SUFFICIENT CONDITION FOR 1-REGULARITY OF A POINT EXTENSION

3.1. Main theorem. The aim of this section is to prove the following statement underlying the combinatorial part in the proof of the main results of this paper.

Theorem 3.1. *Let \mathcal{X} be a homogeneous coherent configuration on n points with indistinguishing number c and maximum valency k . Suppose that $2c(k - 1) < n$, i.e., inequality (3) holds. Then every one point extension of \mathcal{X} is 1-regular. In particular, $b(\mathcal{X}) \leq 2$.*

The proof of Theorem 3.1 will be given in the end of this section. The idea is to deduce the 1-regularity of the point extension \mathcal{X}_α from Lemma 3.6 stating that inequality (3) implies the connectedness of the binary relations s_{max} and s_α defined in Subsection 3.2. Note that this condition itself implies that any pair from s_{max} forms a base of \mathcal{X} (Lemma 3.3).

3.2. Relations s_{max} and s_α . Let $\mathcal{X} = (\Omega, S)$. Recall that $k = k(\mathcal{X})$ is the maximal valency of \mathcal{X} . Denote by s_{max} the union of all relations in the set

$$S_{max} = \{s \in S : n_s = k\}.$$

Then, obviously, $S_{max} \subset S$ and $s_{max} \in S^\cup$. Moreover, since \mathcal{X} is homogeneous, we have $n_{s^*} = n_s$ for all $s \in S$, and hence, the relation s_{max} is symmetric. We are interested in the connectedness of it, i.e., the connectedness of the graph with vertex set Ω and edge set s_{max} . Note that, in general, this graph is not connected: take \mathcal{X} to be the homogeneous coherent configuration of rank 4 that is associated with a finite projective plane.

With any point $\alpha \in \Omega$, we associate a binary relation $s_\alpha \subseteq \alpha s_{max} \times \alpha s_{max}$ that consists of all pairs (β, γ) such that the colored triangle $\{\alpha, \beta, \gamma\}$ is uniquely determined by the side colors $r = r(\alpha, \beta)$, $s = r(\beta, \gamma)$ and $t = r(\alpha, \gamma)$, and one of the sides $\{\alpha, \beta\}$ or $\{\alpha, \gamma\}$, see Fig. 1. More precisely,

$$s_\alpha = \{(\beta, \gamma) \in \alpha s_{max} \times \alpha s_{max} : c_{rs}^t = 1\},$$

This relation is symmetric. Indeed, we have $n_t = n_r = k$. Since also $n_{r^*} = n_r$, it follows from (4) that $n_t c_{rs}^t = n_{r^*} c_{st^*}^{r^*} = n_r c_{ts^*}^r$. This implies that $c_{ts^*}^r = c_{rs}^t = 1$, and hence, $(\gamma, \beta) \in s_\alpha$.

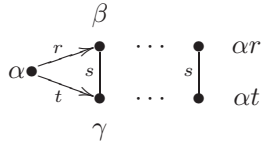


FIGURE 1. A part of the relation s_α .

Lemma 3.2. *Suppose that the graph s_α is connected. Denote by T_α the set of all basis relations of the coherent configuration \mathcal{X}_α that are contained in $\alpha s_{max} \times \alpha s_{max}$. Then*

$$(15) \quad |\beta t| = 1 \quad \text{for all } t \in T_\alpha, \beta \in \alpha s_{max}.$$

Proof. One can see that the set αs_{max} is the union of fibers of \mathcal{X}_α (see [17, Lemma 2.2]). Therefore,

$$(16) \quad \alpha s_{max} \times \alpha s_{max} = \bigcup_{t \in T_\alpha} t.$$

Let $t \in T_\alpha$ and $\beta \in \alpha s_{max}$. Then $\beta \in \alpha r$ for some $r \in S_{max}$. In view of (16), there exists a point $\beta' \in \alpha s_{max}$ contained in βt . By the connectedness of s_α , there exists a path P in s_α connecting β and β' . If this path has length $l = 1$, then by the definition of s_α , we have $c_{rt'}^s = 1$, where $t' \in S$ and $s \in S_{max}$ are unique relations such that $t \subset t'$ and $\beta' \in \alpha s$, respectively. Then, obviously, $\{\beta'\} = \beta t$, as required.

Suppose that $l \geq 2$. Note that if β_1, β_2 , and β_3 are successive vertices of P , then they belong to αs_{max} and

$$\{\beta_2\} = \beta_1 t_1, \quad \{\beta_3\} = \beta_2 t_2,$$

where t_1 and t_2 are the basis relations of \mathcal{X}_α that contain (β_1, β_2) and (β_2, β_3) , respectively. In particular, $t_1, t_2 \in T_\alpha$ and

$$\{\beta_3\} = \beta_1 t_3,$$

where t_3 is a unique relation of T_α containing the pair (β_1, β_3) . This proves the required statement for $l = 2$, and, hence, for all positive integers l by induction. ■

Lemma 3.3. *If s_{max} and all s_α , $\alpha \in \Omega$, are connected relations, then $\{\alpha, \beta\}$ is a base of the coherent configuration \mathcal{X} for each $\beta \in \Omega$ such that $(\alpha, \beta) \in s_{max}$.*

Proof. Let $\alpha \in \Omega$ and $\beta \in \alpha s_{max}$. Denote by Γ the set of all points $\gamma \in \Omega$ for which the singleton $\{\gamma\}$ is a fiber of the coherent configuration $\mathcal{X}_{\alpha, \beta}$. Then obviously $\alpha, \beta \in \Gamma$. We claim that

$$(17) \quad \gamma s_{max} \subseteq \Gamma \quad \text{or} \quad \gamma s_{max} \cap \Gamma = \emptyset$$

for all $\gamma \in \Gamma$. Indeed, suppose on the contrary that there exist points $\gamma \in \Gamma$ and $\gamma_1, \gamma_2 \in \gamma s_{max}$ such that $\gamma_1 \in \Gamma$ and $\gamma_2 \notin \Gamma$. Since s_γ is a connected relation, there is an s_γ -path connecting γ_1 and γ_2 . Moreover, the definition of s_γ implies that if some point in this path is inside Γ , then the next point in this path must be also inside Γ . Therefore $\gamma_2 \in \Gamma$, a contradiction.

Denote by Γ_0 the set of all points $\gamma \in \Gamma$ with $\gamma s_{max} \subseteq \Gamma$. Then $\alpha \in \Gamma_0$, because, as it follows from (17), the set αs_{max} contains $\beta \in \Gamma$. Therefore, Γ_0 contains the connected component of s_{max} that contains α . Since s_{max} is connected, this implies that $\Gamma_0 = \Omega$, and hence $\Gamma = \Omega$. By the definition of Γ , this means that any fiber of the coherent configuration $\mathcal{X}_{\alpha, \beta}$ is a singleton. Thus, $\{\alpha, \beta\}$ is a base of \mathcal{X} . ■

3.3. Connected components of s_α . One can treat s_α also as the graph with vertex set αs_{max} and edge set s_α . The set of all connected components of this graph that contain a vertex in αu for fixed $u \in S_{max}$ is denoted by $C_\alpha(u) = C(u)$.

Lemma 3.4. *Let $u, v \in S_{max}$. Suppose that $C(u) \cap C(v) \neq \emptyset$. Then $C(u) = C(v)$ and $|\alpha u \cap C| = |\alpha v \cap C|$ for all $C \in C(u)$.*

Proof. Let $C_0 \in C(u) \cap C(v)$. Then C_0 contains vertices $\beta \in \alpha u$ and $\gamma \in \alpha v$ connected by an s_α -path, say

$$\beta = \beta_0, \beta_1, \dots, \beta_m = \gamma,$$

where $(\beta_i, \beta_{i+1}) \in s_\alpha$ for $i = 0, \dots, m-1$. By the definition of s_α , this implies that

$$(18) \quad c_{u_i v_i}^{u_{i+1}} = 1$$

for all i , where $u_i = r(\alpha, \beta_i)$ and $v_i = r(\beta_i, \beta_{i+1})$. Therefore, it is easily seen that for every $C \in C(u)$ given a vertex $\beta' \in C$, there is a unique s_α -path

$$\beta' = \beta'_0, \beta'_1, \dots, \beta'_m = \gamma'$$

such that $\gamma' \in \alpha v$ and $r(\alpha, \beta'_i) = u_i$ and $r(\beta'_i, \beta'_{i+1}) = v_i$ for all i . In view of (18), no vertices β_i and β'_i coincide whenever $\beta \neq \beta'$. Thus, the mapping

$$\alpha u \rightarrow \alpha v, \beta' \mapsto \gamma'$$

is a bijection. Obviously, the vertex γ' belongs to the component C of the graph s_α that contains β' . Since this is true for all $\beta' \in C$ and all $C \in C(u)$, the required statement follows. \blacksquare

For a relation $u \in S_{max}$ and a point $\delta \in \Omega$, denote by $p_u(\delta)$ the number of pairs $(\beta, \gamma) \in \alpha u \times \alpha u$ such that $\beta \neq \gamma$ and $r(\beta, \delta) = r(\gamma, \delta)$. Here, $|\alpha u| = n_u = k$. Therefore, αu contains exactly $k(k-1)$ pairs of distinct elements. Now we are able to estimate from above the sum of $p_u(\delta)$ in terms of the indistinguishing numbers of the corresponding basis relations $c(r(\beta, \gamma))$ as well as the indistinguishing number c of \mathcal{X} . Indeed,

$$(19) \quad k(k-1)c \geq \sum_{\beta, \gamma} c(r(\beta, \gamma)) \geq \sum_{\delta \in \Delta} p_u(\delta)$$

for any set $\Delta \subseteq \Omega$. On the other hand, the number $p_u(\delta)$ can be computed by means of the intersection numbers. Namely, if $v = r(\alpha, \delta)$, then, obviously,

$$(20) \quad p_u(\delta) = \sum_{w \in T_{u,v}} c_{uw}^v (c_{uw}^v - 1)$$

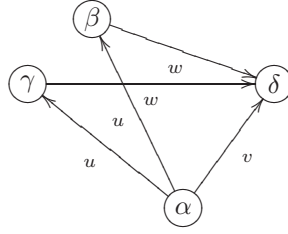
where $T_{u,v} = \{w \in u^*v : c_{uw}^v > 1\}$ (see Fig 2). In particular, the number $p_u(\delta)$ does not depend on $\delta \in \alpha v$.

Lemma 3.5. *In the above notation, the following statements hold:*

- (1) *if either $n_u > n_v$, or $n_u = n_v$ and $C(u) \neq C(v)$, then $p_u(\delta) \geq k$,*
- (2) *if $n_u = n_v$, $C(u) = C(v)$, and $|C(u)| > 1$, then $p_u(\delta) \geq k/2$.*

Proof. To prove statement (1), suppose that either $n_u > n_v$, or $n_u = n_v$ and $C(u) \neq C(v)$. Then

$$(21) \quad T_{u,v} = u^*v.$$

FIGURE 2. A relation $w \in T_{u,v}$ with $c_{uv}^w > 1$.

Indeed, obviously, $T_{u,v} \subseteq u^*v$. The converse inclusion is true if $n_u > n_v$, because in this case, $c_{uw}^v = n_u c_{v^*u}^w / n_v > 1$ for all $w \in u^*v$. Let now $C(u) \neq C(v)$. Then the sets $C(u)$ and $C(v)$ are disjoint (Lemma 3.4). This implies that if $\beta \in \alpha u$ and $\gamma \in \alpha v$, then $(\beta, \gamma) \notin s_\alpha$. Therefore, $c_{uw}^v > 1$ for all $w \in u^*v$, whence again $u^*v \subseteq T_{u,v}$. Thus relation (21) is completely proved. Together with (20), this shows that

$$p_u(\delta) = \sum_{w \in T_{u,v}} c_{uw}^v (c_{uw}^v - 1) \geq \sum_{w \in T_{u,v}} c_{uw}^v = \sum_{w \in u^*v} c_{uw}^v = n_u = k,$$

as required. Observe that the penultimate equality is the well-known identity for homogenous coherent configurations.

To prove statement (2), suppose that $n_u = n_v$, $C(u) = C(v)$, and $|C(u)| > 1$. Let us choose $C \in C(u)$ so that the number $|\alpha u \cap C|$ is minimum possible. Then

$$(22) \quad |\alpha u \setminus C| \geq k/2,$$

because $|C(u)| > 1$ and $|\alpha u| = n_u = k$. Next, since $C(u) = C(v)$, we have $C \in C(v)$. Moreover, αv is not contained in C , because $|C(v)| = |C(u)| > 1$. Since $p_u(\delta)$ does not depend on the choice of $\delta \in \alpha v$, we may assume that $\delta \in \alpha v \cap C$. Then no point $\beta \in \alpha u \setminus C$ belongs to the component of s_α that contains δ . In particular, (δ, β) is not an edge of s_α . Therefore,

$$c_{uw}^v > 1 \quad \text{for all } w \in T,$$

where T is the set of all $w = r(\beta, \delta)$ with $\beta \in \alpha u \setminus C$. By (20) and (22), we obtain

$$p_u(\delta) = \sum_{w \in T_{u,v}} c_{uw}^v (c_{uw}^v - 1) \geq \sum_{w \in T} c_{uw}^v = |\alpha u \cap \delta T^*| \geq |\alpha u \setminus C| \geq k/2,$$

as required. ■

3.4. The connectedness of s_{max} and s_α . Using Lemmas 3.4 and 3.5, we will prove that the hypothesis of Theorem 3.1 gives a sufficient condition for the graphs s_α and s_{max} to be connected. Note that by Lemma 3.3, this establishes the second statement of Theorem 3.1.

Lemma 3.6. *Suppose that $2c(k-1) < n$ and $k \geq 2$. Then the graphs s_α and s_{max} are connected. Moreover, $|\alpha s_{max}| > n/2$.*

Proof. To prove the first statement, we claim that

$$(23) \quad |C(u)| = 1 \quad \text{for all } u \in S_{max}.$$

Indeed, if this is not true, then there exists $u \in S_{max}$ such that $|C(u)| \geq 2$. Lemma 3.5 yields that $p_u(\delta) \geq k/2$ for all points $\delta \in \Omega$. By (19) with $\Delta = \Omega$, this implies that

$$c \geq \frac{1}{k(k-1)} \sum_{\delta \in \Omega} p_u(\delta) \geq \frac{1}{k(k-1)} \frac{|\Omega|k}{2} = \frac{n}{2(k-1)},$$

which contradicts the lemma hypothesis. Thus, formula (23) is proved.

Suppose on the contrary that the graph s_α is not connected for some $\alpha \in \Omega$. Then it has a component C containing at most half of the vertices, that is

$$(24) \quad 2|C| \leq |\alpha s_{max}| < n.$$

By (23), one can find a relation $u \in S_{max}$ such that $C(u) = C$. Then for any point $\delta \in \Omega \setminus C$, we have $n_v < n_u$ or $C(v) \neq C(u)$, where $v = r(\alpha, \delta)$ (if $C(v) = C(u) = C$, then $\delta \in C$). By statement (1) of Lemma 3.5, this implies that $p_u(\delta) \geq k$. On the other hand, $2|\Omega \setminus C| \geq n$ by (24). From (19) with $\Delta = \Omega \setminus C$, we obtain that

$$(25) \quad c \geq \frac{1}{k(k-1)} \sum_{\delta \in \Omega \setminus C} p_u(\delta) \geq \frac{1}{k(k-1)} |\Omega \setminus C|k \geq \frac{n}{2(k-1)},$$

which contradicts the lemma hypothesis. Thus, the graph s_α is connected.

To prove that the graph s_{max} is also connected, suppose on the contrary that one of its components, say C , has at most $n/2$ points. Let $\alpha \in C$ and $u \in S_{max}$. Then $\alpha u \subseteq C$ and $n_u > n_v$ for all $v = r(\alpha, \delta)$ with $\delta \in \Omega \setminus C$. By statement (1) of Lemma 3.5, this implies that $p_u(\delta) \geq k$ for all such δ . Again inequality (25) hold, which contradicts the lemma hypothesis.

To prove the second statement, denote by V the union of all $v \in S$ with $n_v < k$, and fix $u \in S_{max}$. Then by statement (1) of Lemma 3.5, we have $p_u(\delta) \geq k$ for all $\delta \in \Omega$ such that $r(\alpha, \delta) \in V$. This implies that

$$\sum_{\delta \in \alpha V} p_u(\delta) \geq k|\alpha V|.$$

Since there are $k(k-1)$ pairs of different points in αu , at least for one of such pairs, say (α, β) , we obtain

$$c \geq c(u) = |\Omega_{\alpha, \beta}| \geq \frac{1}{k(k-1)} \sum_{\delta \in \alpha V} p_u(\delta) \geq \frac{k|\alpha V|}{k(k-1)} = \frac{|\alpha V|}{k-1}.$$

By the lemma hypothesis, this implies that $2\frac{|\alpha V|}{k-1}(k-1) < n$, whence $|\alpha V| < n/2$. Since $|\alpha V| + |\alpha s_{max}| = n$, we are done. \blacksquare

3.5. Proof of Theorem 3.1. Obviously, we may assume that $k \geq 2$. Let $\alpha \in \Omega$. It suffices to prove that each $\beta \in \alpha s_{max}$ is a regular point of the coherent configuration \mathcal{X}_α . Suppose on the contrary that β is not a regular point of \mathcal{X}_α . Then this coherent configuration has a basis relation v such that $|\beta v| \geq 2$. Therefore, there exist distinct points γ_1 and γ_2 such that

$$(26) \quad r(\beta, \gamma_1) = v = r(\beta, \gamma_2).$$

Let T_α be the set defined in Lemma 3.2. Then in view of (15), $v \notin T_\alpha$. Therefore, neither γ_1 nor γ_2 belongs to αs_{max} .

Let us verify that formula (26) holds with β replaced by arbitrary $\beta' \in \alpha_{S_{max}}$ and suitable distinct points γ_1 and γ_2 . By virtue of (15), the relation $u \in T_\alpha$ containing the pair (β, β') is of the form

$$u = \{(\delta, \delta^g) : \delta \in \Delta\}$$

for some bijection $g : \Delta \rightarrow \Delta'$, where Δ and Δ' are the fibers of \mathcal{X}_α containing β and β' , respectively. Define a permutation $f \in \text{Sym}(\Omega)$ by

$$\omega^f = \begin{cases} \omega^g & \text{if } \omega \in \Delta, \\ \omega^{g^{-1}} & \text{if } \omega \in \Delta', \\ \omega & \text{otherwise.} \end{cases}$$

Then the graph of f is the union of basis relations of \mathcal{X}_α , each of which is of valency 1. One can see that in this case, f takes any basis relation of \mathcal{X}_α to another basis relation. This implies that f is an isomorphism of the coherent configuration \mathcal{X}_α to itself. Therefore, by equality (26), we have

$$r(\beta', \gamma_1) = r(\beta^f, \gamma_1^f) = r(\beta^f, \gamma_2^f) = r(\beta', \gamma_2),$$

and the claim is proved. Thus, the set $\Omega_{\gamma_1, \gamma_2}$ contains $\alpha_{S_{max}}$. By the theorem hypothesis and second statement of Lemma 3.6, this implies that

$$c \geq |\Omega_{\gamma_1, \gamma_2}| \geq |\alpha_{S_{max}}| > n/2.$$

However, then $n > 2c(k-1) > n(k-1)$, which is impossible for $k > 1$. \blacksquare

4. INEQUALITY (3) IN SIMPLE GROUPS OF LIE TYPE

The main purpose of the following two sections is to prove Theorem 4.1 below, from which Theorems 1.2 and 1.3 were deduced in Introduction. In this section, we reduce the proof to Lemma 4.3, which will be proved in the next section.

Theorem 4.1. *For the Cartan scheme \mathcal{X} of every group $G \in \mathcal{L}$, inequality (3) holds.*

We generally follow the notation of well-known Carter's book [6] with some exceptions that we explain below inside parentheses. If Φ_l is a simple Lie algebra of rank l , then $\Phi_l(q)$ is the simple Chevalley group of rank l over a field of order q . Let B , N , and $H = B \cap N$ be a Borel, monomial, and Cartan subgroups of a simple Chevalley group $\Phi_l(q)$ as in [6], while $W = N/H$ be the corresponding Weil group. Then [6, Proposition 8.2.1] implies that the subgroups B and N form a (B, N) -pair of $\Phi_l(q)$. If τ is a symmetry of the Dynkin diagram of Φ_l of order t (and the corresponding automorphism of $\Phi_l(q^t)$), then ${}^t\Phi_l(q)$ is the simple twisted group of Lie type (in [6] such a group is denoted as ${}^t\Phi_l(q^t)$). Again B , N , and $H = B \cap N$ stand for Borel, monomial, and Cartan subgroups of a simple twisted group of Lie type, and $W = N/H$ is the Weil group (in [6] they are denoted by B^1 , N^1 and so on). It follows from [6, Theorem 13.5.4] that in this case B and N form a (B, N) -pair of ${}^t\Phi_l(q)$ again. In the sake of brevity we will use notation ${}^t\Phi_l(q)$ for all simple groups of Lie type, assuming that t is the empty symbol in the case of untwisted groups. Recall also that the order w of the Weil group W does not depend on the order of the underlying field.

Let G be a finite simple group of Lie type, and let $\mathcal{X} = (\Omega, S)$ be the Cartan scheme of G , where the corresponding (B, N) -pair is as in the previous paragraph

(see Definition 1.1). In particular, $\Omega = G/H$ and $S = \text{Orb}(G, \Omega^2)$. Put $n = |\Omega|$, $k = k(\mathcal{X})$, and $c = c(\mathcal{X})$.

Lemma 4.2. *There exists an element $g_0 \in G$ such that $H \cap H^{g_0} = 1$. In particular,*

$$(27) \quad k = \max_{s \in S} n_s = |H|.$$

Proof. It is well known that $B = U \rtimes H$ is the semidirect product of U and H , where U is the unipotent radical of B . By [7, Propositions 5.1.5 and 5.1.7], there exists a unipotent element $u \in U$ such that $C_G(u) \leq U$ (in fact, u can be chosen as a regular unipotent element fixed by an appropriate Frobenius map of the corresponding algebraic group), it follows that $[h, u] \neq 1$ for every $h \in H^\#$. On the other hand, if $h \in H \cap H^u$, then $[h, u] \in H \cap U = 1$, so $h = 1$ and $g_0 = u$ is the desired element of G .² Now, in view of (6),

$$k = \max_{s \in S} n_s = \max_{g \in G} \frac{|H|}{|H \cap H^g|} = \frac{|H|}{|H \cap H^{g_0}|} = |H|. \blacksquare$$

Observe that G satisfies the hypotheses of Lemmas 2.3 and 2.4. Indeed, the transitivity of the action of G on Ω is evident, while the monomial subgroup N of G satisfies the additional condition from Lemma 2.4 due to [6, Proposition 8.4.5]. This enables us to estimate the indistinguishing number c and prove the required inequality (3). We need the following lemma whose strictly group-theoretic proof is postponed to the next section.

Below, for a coset $\overline{y} = Hy$, set $M_{\overline{y}} = \{u \in \overline{y} : u^G \cap H \neq \emptyset\}$. Note that all elements of $M_{\overline{y}}$ are semisimple. For an integer m , set

$$M_{\overline{y},m} = \{u \in M_{\overline{y}} : |u^G| \geq m\}, \quad \text{and} \quad M'_{\overline{y},m} = M_{\overline{y}} \setminus M_{\overline{y},m}.$$

Put

$$r_m = \max_{y \in G \setminus H} |M'_{\overline{y},m}| \quad \text{and} \quad m_0 = \min_{\emptyset \neq y^G \cap H \neq y^G} |y^G|.$$

Lemma 4.3. *In the above notation, there exists a positive integer m such that*

$$(28) \quad \frac{k}{m} + \frac{r_m}{m_0} \leq \frac{1}{2wk}$$

for all groups $G \in \mathcal{L}$ and every coset $\overline{y} \neq H$.

Proof of Theorem 4.1. Immediately follows from Lemma 4.3 and Lemma 4.4 below.

Lemma 4.4. *Let G be a simple group of Lie type. Suppose that there exists an integer m such that inequality (28) holds. Then for the Cartan scheme \mathcal{X} of G , inequality (3) is satisfied.*

Proof. It follows from Lemma 2.3 that

$$(29) \quad c \leq \max_{y \in G \setminus H} \sum_{h \in H} \chi(hy) \leq \max_{y \in G \setminus H} \sum_{x \in M_{\overline{y}}} \chi(x).$$

Let $x \in M_{\overline{y}}$. Then by Lemma 2.4, $\text{Fix}(x) \neq \emptyset$, i.e. there are $h_0 \in H$ and $g_0 \in G$ with $x = h_0^{g_0}$. In the notation of that lemma $\chi(x) = |N : (C \cap N)| |\Omega|/|x^G|$.

²The alternative way to establish the same is to apply Zenkov's theorem [22]. It yields that since H is abelian, there is an element $g_0 \in G$ such that $H \cap H^{g_0}$ lies in the Fitting subgroup of G , which is trivial in the case of a simple group G .

Furthermore, $|N : (C \cap N)| \leq |N/H| = |W| = w$, because $H \leq C \cap N$. We conclude that

$$(30) \quad \chi(x) \leq \frac{wn}{|x^G|}.$$

Let m be a positive integer from Lemma 4.3. Our definitions imply that $|x^G| \geq m$ for all $x \in M_{\overline{y},m}$ and $|x^G| \geq m_0$ for all $x \in M_{\overline{y}}$. Taking into account $|M_{\overline{y}}| \leq |H| = k$ and formula (30), we obtain

$$\begin{aligned} \sum_{x \in M_{\overline{y}}} \chi(x) &\leq wn \sum_{x \in M_{\overline{y}}} \frac{1}{|x^G|} \leq wn \left(\sum_{x \in M_{\overline{y},m}} \frac{1}{|x^G|} + \sum_{x \in M_{\overline{y},m}'} \frac{1}{|x^G|} \right) \leq \\ &wn \left(\frac{|M_{\overline{y},m}|}{m} + \frac{|M_{\overline{y},m}'|}{m_0} \right) \leq wn \left(\frac{k}{m} + \frac{r_m}{m_0} \right). \end{aligned}$$

By inequality (28), this implies

$$\sum_{x \in M_{\overline{y}}} \chi(x) \leq wn \cdot \frac{1}{2wk} = \frac{n}{2k}$$

for any $y \in G \setminus H$. In view of (29), this immediately shows that $2ck \leq n$, as required. \blacksquare

5. PROOF OF LEMMA 4.3

We begin with a simple remark. Suppose that \overline{G} is a central cover of G , i.e., $G \cong \overline{G}/Z(\overline{G})$, and subgroups \overline{B} and \overline{N} , whose images in G are B and N , form the (B, N) -pair of \overline{G} . If $Z(\overline{G}) \leq \overline{H} = \overline{B} \cap \overline{N}$, then we may exploit the action of \overline{G} on its Cartan subgroup \overline{H} instead of the action of G on H , because

$$\text{Inv}(G, G/H) = \text{Inv}(\overline{G}, \overline{G}/\overline{H}).$$

Observe that the values of n , w , the sizes of conjugacy classes remain the same, and $k = |H|$ does not exceed the order of $|\overline{H}|$. In particular, we may take as \overline{G} the universal cover \widehat{G} of G or, in the case of classical groups, the group of linear transformation \widetilde{G} of vector space with an appropriate form, whose quotient by its central subgroup isomorphic to G .

First, suppose that G is a simple exceptional group. We prove that relation (28) holds for $m = m_0$. Note that the size of the conjugacy class of a semisimple element of G can be estimated from below by means of results in [8, 9] (for all exceptional groups other than the Ree and Suzuki groups, this was done in [20]). The corresponding low bounds for m_0 are listed in the second column of Table 1. The values from the third and forth columns are well-known. Using this table, one can easily check that $m_0 \geq 2w|\widehat{H}|^2$, whence

$$(31) \quad m_0 \geq 2wk^2,$$

which is equivalent to inequality (28) for $m = m_0$, because $r_{m_0} = 0$ in this case.

TABLE 1.

${}^t\Phi_l$	m_0	$ \tilde{H} $	$ W $
E_8	q^{112}	$(q-1)^8$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
E_7	$(1/2)q^{64}$	$(q-1)^7$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
E_6	$(1/3)q^{30}$	$(q-1)^6$	$2^7 \cdot 3^4 \cdot 5$
2E_6	$(1/3)q^{30}$	$(q-1)^4(q+1)^2$	$2^7 \cdot 3^2$
F_4	q^{16}	$(q-1)^4$	$2^7 \cdot 3^2$
G_2	$q^4(q^3-1)$	$(q-1)^2$	$2^2 \cdot 3$
3D_4	q^{16}	$(q-1)(q^3-1)$	$2^2 \cdot 3$
2F_4	$q^6(q-1)(q^3+1)$	$(q-1)^2$	2^4
2G_2	$q^2(q^2+q+1)$	$q-1$	2
2B_2	$q^2(q-1)$	$q-1$	2

Now G is a simple classical group. Our main source to estimate the size of a conjugacy class of G is [5].³ Let V be a natural module over a field \mathbb{F}_{q^v} , where $v = 2$ in the case of unitary groups and $v = 1$ otherwise, such that $G \leq PSL(V)$, and let \tilde{G} be preimage of G in $SL(V)$. If $x \in G$ and $X \leq G$, then \tilde{x} and \tilde{X} are preimages of x and X in \tilde{G} . We also agree to fix the base of V in such a way that the preimage \tilde{H} of the Cartan subgroup H consists of diagonal matrices. Following [5, Definition 3.16], for an element $x \in G$ we denote by $\nu(x)$ the codimension of largest eigenspace of \tilde{x} on $\overline{V} = V \otimes K$, where K is algebraic closure of \mathbb{F}_q . For elements x conjugated to elements of H , which are of the prime interest for our purposes, $\nu(x)$ is just equal to the difference between the dimension of V and the maximum eigenvalue multiplicity of the diagonal matrix \tilde{h} with $x = h^q$. We gather in Table 2 the lower bounds m_0 on the sizes of conjugacy classes from [5, Table 3.7-3.9] as well as the numbers $|W|$ and upper bounds for $|\tilde{H}|$.⁴ This table also contains the numbers l_0 and a defining the class \mathcal{L} . We also suppose that q is odd in the case of the groups $B_l(q)$ due to the well-known isomorphism $B_l(q) \cong C_l(q)$ for even q .

Now we are ready to define the number m for simple classical groups. Denote by m_1 the low bound for $|x^G|$ with $\nu(x) \geq 2$ that was found in [5, Tables 3.7-3.9]; the relevant values of m_1 are collected in the third column of Table 3. Set

$$(32) \quad m = \begin{cases} m_0, & \text{if } \nu(h) \geq 2 \text{ for all } h \in H^\#, \\ m_1, & \text{otherwise.} \end{cases}$$

Note that for any coset $\overline{y} = Hy$ distinct from H , $M_{\overline{y},m} = \{x \in M_{\overline{y}} \mid \nu(x) \geq 2\}$ and $M'_{\overline{y},m}$ consist of all elements $x \in M_{\overline{y}}$ such that $\nu(x) = 1$. Recall that $r_m = \max_{y \in G \setminus H} |M'_{\overline{y},m}|$.

Lemma 5.1. *In the above notation the following statements hold.*

³It is worth noting that in ‘an asymptotical sense’ the required lower bounds can be taken from [14, Lemma 3.4]. We chose to use results from the later paper [5] in order to obtain the numerical values of l_0 and a .

⁴It is worth mentioning that despite [5, Tables 3.7-3.9] contain the bounds on the sizes of conjugacy classes in the group $\text{Inndiag}(G)$ rather than G itself, we get the correct bounds, because for $h \in H$ we obviously have $|G : C_G(h)| = |\text{Inndiag}(G) : C_{\text{Inndiag}(G)}(h)|$.

TABLE 2.

${}^t\Phi_l$	conditions	m_0	$ \tilde{H} $	$ W $	l_0	a
A_l		$q^{2l}/2$	$(q-1)^l$	$(l+1)!$	7	4
2A_l	l odd	$\frac{q^{4l-3}}{2(q+1)}$	$(q-1)^{\lfloor \frac{l+1}{2} \rfloor} (q+1)^{\lfloor \frac{l}{2} \rfloor}$	$2^{\lfloor \frac{l+1}{2} \rfloor} \lfloor \frac{l+1}{2} \rfloor!$	6	4
2A_l	l even	$\frac{q^{2l+1}}{2(q+1)}$	$(q-1)^{\lfloor \frac{l+1}{2} \rfloor} (q+1)^{\lfloor \frac{l}{2} \rfloor}$	$2^{\lfloor \frac{l+1}{2} \rfloor} \lfloor \frac{l+1}{2} \rfloor!$	6	4
B_l	$\frac{l(q-1)}{2}$ odd	$\frac{q^{4l-1}}{4(q+1)}$	$\frac{(q-1)^l}{2}$	$2^l l!$	4	4
B_l	$\frac{l(q-1)}{2}$ even	$\frac{q^{2l+1}}{4(q+1)}$	$\frac{(q-1)^l}{2}$	$2^l l!$	4	4
C_l		$\frac{q^{4l-4}}{2}$	$(q-1)^l$	$2^l l!$	3	4
D_l		$\frac{q^{4l-3}}{4(q+1)}$	$(q-1)^l$	$2^{l-1} l!$	4	2
2D_l		$\frac{q^{4l-3}}{4(q+1)}$	$(q-1)^{l-1} (q+1)$	$2^{l-1} l!$	4	2

TABLE 3.

${}^t\Phi_l$	conditions	m_1	r_m
A_l		$\frac{q^{4(l-1)}}{2}$	$\frac{l(l+1)(q-1)^2}{2} - 1$
2A_l	l even	$\frac{q^{4l-3}}{2(q+1)}$	$\frac{(l+1)(q+1)^2}{2} + q$
B_l	$\frac{l(q-1)}{2}$ even	$\frac{q^{4l-1}}{4(q+1)}$	$\frac{l(q-3)}{2} + 1$

- (1) If G is one of the groups A_l , 2A_l with l even, and B_l with $l(q-1)/2$ even, then the number r_m does not exceed the number in the fourth column of the corresponding row of Table 3.
- (2) If G is one of the other simple classical groups, then $\nu(h) \geq 2$ for every $h \in H^\#$.

Proof. It is well known and easily verified that the diagonal subgroup of a classical matrix group contains an element h with $\nu(h) = 1$ only if G is one of the groups in statement (1). Therefore, we need only to estimate r_m in these cases.

We claim that in any case

$$(33) \quad r_m \leq |\{h \in H^\# : \nu(h) \leq 2\}| =: u.$$

Indeed, let y and $z = hy$ be distinct elements of $M_{y,m}^L$. Then $\nu(y) = \nu(z) = 1$. Therefore, each of the matrices \tilde{y} and \tilde{z} has an eigenvalue of multiplicity $\dim(V) - 1$. Since these matrices are conjugated to diagonal matrices, this implies that the matrix $\tilde{h} = \tilde{y}(\tilde{z})^{-1}$ has an eigenvalue of multiplicity at least $\dim(V) - 2$. Therefore, $\nu(h) \leq 2$. Since this is true for all $z \in M_{x,m}^L$, we are done.

Let $G = A_l$. Then the required statement immediately follows from (33) by a direct calculation of the number u , which is equal the number of diagonal matrices in $\text{SL}(l+1, q)$ with at most 2 distinct diagonal entries.

Suppose that $G = {}^2A_l$ and l is even. To check the required upper bound on the number u , we observe that a base of V can be chosen so that any matrix $\tilde{h} \in \tilde{H}$ is of the form

$$\tilde{h} = \text{diag}(\lambda_1, \dots, \lambda_r, \lambda_0, \lambda_1^{-q}, \dots, \lambda_r^{-q}),$$

where $r = l/2$, $\lambda_i \in \mathbb{F}_{q^2}$ for all i , $(\lambda_0)^{q+1} = 1$, and $\lambda_0(\lambda_1)^{1-q} \cdots (\lambda_r)^{1-q} = 1$. If, in addition, $\nu(h) \leq 2$ and $l \geq 6$, then either

$$\tilde{h} = \text{diag}(\lambda, \dots, \lambda, \lambda_0, \lambda, \dots, \lambda),$$

where $\lambda^{q+1} = 1$ and $\lambda_0 \lambda^l = 1$, or

$$\tilde{h} = \text{diag}(\lambda, \dots, \lambda, \mu, \lambda, \dots, \lambda, \lambda_0, \lambda, \dots, \lambda, \mu^{-q}, \lambda, \dots, \lambda),$$

where $\lambda = \lambda_0$, $\lambda^{l-1} \mu^{1-q} = 1$, and μ takes an arbitrary j -th of the first r positions (so μ^{-q} takes the $(r+1+j)$ -th position). The rest is routine.

Let $G = B_l$ and $l(q-1)/2$ even. Then $G = \tilde{G} = \Omega_{2l+1}(q)$. To estimate u from above, choose a base of V so that any matrix $h \in H$ is of the form

$$h = \text{diag}(\xi^{k_1}, \dots, \xi^{k_l}, \xi^{-k_1}, \dots, \xi^{-k_l}, 1),$$

where ξ is a primitive element of the field \mathbb{F}_q , and the number $k_1 + \dots + k_l$ is even. If, in addition, $\nu(h) \leq 2$ and $l \geq 3$, then either

$$h = \text{diag}(-1, \dots, -1, 1)$$

(recall that $l(q-1)/2$ is even), or

$$h = \text{diag}(1, \dots, 1, \mu, 1, \dots, 1, \mu^{-1}, 1, \dots, 1),$$

where μ is a nonzero square in \mathbb{F}_q and takes an arbitrary j -th of the first l positions (so μ^{-1} takes the $(l+j)$ -th position). Thus, $u \leq l(q-3)/2 + 1$. ■

To complete the proof, we verify inequality (28) for the number m defined by (32). Observe that, due to (32) and Lemma 5.1, the number r_m equals 0 in all cases when $m = m_0$. In the latter case, it suffices to verify inequality (31). We proceed further case by case.

Let $G = C_l(q)$. Here $m = m_0$ and we need to prove that $m_0 \geq 2wk^2$. According to Table 2, it means that for $l \geq 3$ and $q \geq 4l$, the following inequality have to be true:

$$\frac{q^{4l-4}}{2} \geq 2^{l+1} l! (q-1)^{2l}.$$

For $l = 3$, this inequality is straightforward. If $l \geq 4$, then $q^{2l-4} \geq 4(2l)^l \geq 2^{l+2} l!$, and we are done.

Let $G = D_l^\varepsilon(q)$. Then $m = m_0$ and to verify (31), we check that

$$\frac{q^{4l-3}}{4(q+1)} \geq 2^l l! (q-1)^{2l-2} (q+1)^2$$

for $l \geq 4$ and $q \geq 2l$. Since $(q-1)^{2l-2} (q+1)^3 < q^{2l+1}$ for all these l and q , it suffices to check that $q^{2l-4} \geq 2^{l+2} l!$. For $l = 4$ it can be verified directly, while for $l > 4$ we have $q^{2l-4} \geq 4(2l)^l \geq 2^{l+2} l!$.

Let $G = A_l(q)$ and suppose that $l \geq 7$ and $q \geq 4l$. By Lemma 5.1, we obtain that

$$r_m \leq \frac{l(l+1)(q-1)^2}{2} - 1 \leq \frac{q^4}{32}.$$

Thus, the left-hand side of (28) can be estimated as follows:

$$(34) \quad \frac{k}{m} + \frac{r_m}{m_0} \leq \frac{2(q-1)^l}{q^{4l-1}} + \frac{2q^4}{32q^{2l}} \leq \frac{2}{q^{4l-3}} + \frac{1}{16q^{2l-4}} \leq \frac{1}{8q^{2l-4}}.$$

On the other hand, $(l+1)! \leq 4^{l-3}l^{l-4}$ for $l \geq 7$: this is verified directly for $7 \leq l \leq 9$, and follows from the obvious inequalities $(l+1)! < l^l < 4^{l-3}l^{l-4}$ for $l \geq 10$. Therefore, in our case, we get the following low bound for the right-hand side of (28):

$$(35) \quad \frac{1}{2wk} = \frac{1}{2(q-1)^l(l+1)!} \geq \frac{1}{2q^l 4^{l-3}l^{l-4}} \geq \frac{1}{2q^l 4q^{l-4}} = \frac{1}{8q^{2l-4}}.$$

Thus, the required statement follows from (34) and (35).

For each of the remaining two series of classical groups the expression on the left-hand side of (28) for $m = m_0 \leq m_1$ does not exceed the same expression for $m = m_1$ (see Tables 2 and 3). Since the expression on the right-hand side in both cases does not depend on whether $m = m_0$ or not, it suffices to verify (28) for $G = {}^2A_l(q)$ (resp., $G = B_l(q)$) independently of the oddness of l (resp., $l(q-1)/2$), taking m_0 and m as in case of even l (resp. even $l(q-1)/2$).

Let $G = {}^2A_l(q)$. Suppose that $l \geq 6$ and $q \geq 4l$. Lemma 5.1 yields that

$$r_m \leq \frac{(l+1)(q+1)^2}{2} + q \leq \frac{q^3}{6}.$$

Put $b = \lfloor l+1/2 \rfloor$. Now, the left-hand side and right-hand side of (28) can be estimated as follows:

$$(36) \quad \frac{k}{m} + \frac{r_m}{m_0} \leq \frac{2(q-1)^b(q+1)^{\lfloor \frac{l}{2} \rfloor + 1}}{q^{4l-3}} + \frac{2q^3(q+1)}{6q^{2l+1}} \leq \frac{2q^l(q+1)}{q^{4l-3}} + \frac{q^3(q+1)}{3q^{2l+1}}$$

and

$$(37) \quad \frac{1}{2wk} \geq \frac{1}{2(q-1)^b(q+1)^{\frac{l}{2}} 2^b b!} \geq \frac{1}{2q^l 2^b b!}.$$

By (36) and (37), it suffices to verify that

$$(38) \quad 2^b b! \leq \frac{3q^{2l-3}}{2(q+1)(q^{l-1}+6)}.$$

However, one can easily check that $2(q+1)(q^{l-1}+6) \leq 3q^l$ and $2^b b! \leq q^{l-3}$ for all $q \geq 4l \geq 25$. Therefore, (38) holds, and we are done.

Let $G = B_l(q)$. Suppose that $l \geq 4$ and $q \geq 4l$. By Lemma 5.1, it follows that

$$r_m \leq \frac{l(q-3)}{2} + 1 \leq \frac{q^2}{8}.$$

Now, the left-hand side and right-hand side of (28) can be estimated as follows:

$$\frac{k}{m} + \frac{r_m}{m_0} \leq \frac{4q^l(q+1)}{2q^{4l-1}} + \frac{4q^2(q+1)}{8q^{2l+1}} = \frac{(q+1)(q^l+4)}{2q^{3l-1}}$$

and

$$\frac{1}{2wk} \geq \frac{1}{2^l l! (q-1)^l} \geq \frac{1}{2^l l! q^{l-1} (q-1)}.$$

Thus, it suffices to verify that

$$(q^l+4)2^l l! \leq 2q^{2l-2}.$$

This is straightforward for $l = 4$. Since $q \geq 4l$, the required inequality holds whenever $l! \leq 2^{l-4}l^{l-2}$, which can be directly checked for $5 \leq l \leq 10$. Finally, if $l \geq 11$, then $l! \leq l^l \leq 2^{l-4}l^{l-2}$. This completes the proof of the lemma.

6. PROOF OF THEOREM 1.4

Note that the actions of $\mathrm{PSL}(2, q)$ and $G = \mathrm{SL}(2, q)$ on the cosets of the corresponding Cartan subgroups are equivalent. Thus, without loss of generality, we may assume that $\mathcal{X} = \mathrm{Inv}(G, \Omega)$, where $\Omega = G/H$ with H being the subgroup of diagonal matrices of G . Thus,

$$(39) \quad |G| = q(q+1)(q-1), \quad |H| = q-1, \quad |\Omega| = |G : H| = q^2 + q.$$

First, we study a structure of \mathcal{X} in terms of double H -cosets (see Subsection 2.3).

One can see that the group $N = N_G(H)$ is the disjoint union of two double H -cosets, namely, H and $HiH = Hi$, where

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Denote by s_1 and s_i the basis relations of \mathcal{X} , for which $D_{s_1} = H$ and $D_{s_i} = HiH$ (see (5)). Clearly, $s_1 = 1_\Omega$.

Lemma 6.1. *Let S be the set of basis relations of the coherent configuration \mathcal{X} . Then given $s \in S$, we have*

$$n_s = \begin{cases} 1 & \text{if } s \in \{s_1, s_i\}, \\ q-1 & \text{otherwise.} \end{cases}$$

In particular, $|S| = q+4$ and $|S_{max}| = q+2$.

Proof. It is easy to verify that $H^x \cap H = 1$ for all $x \in G \setminus N$ and $N = H \cup Hi$. Thus, the required statements follow from formula (6). \blacksquare

Denote by U and V the subgroups (in G) of unipotent upper triangular and low triangular matrices, respectively. Since, obviously, $H \leq N_G(U) \cap N_G(V)$, we conclude that

$$(40) \quad HuH = HU^\# = U^\#H \quad \text{and} \quad HvH = HV^\# = V^\#H$$

for all $u \in U^\#$ and $v \in V^\#$. Denote by s_u and s_v the basis relations of \mathcal{X} , for which $D_{s_u} = HuH$ and $D_{s_v} = HvH$, respectively. In view of (40), these relations do not depend on the choice the matrices

$$u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad v = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix},$$

where x and y are nonzero elements of the field \mathbb{F}_q . Clearly, from (40), it follows that $(s_u)^* = s_u$ and $(s_v)^* = s_v$.

Lemma 6.2. *In the above notation, let $s \in S$. Then*

- (1) $c_{s_u s}^{s_v} = 0$ if $s = s_1$ or s_i , and $c_{s_u s}^{s_v} = 1$ otherwise,
- (2) if $s \notin \{s_1, s_i, s_u, s_v\}$, then $c_{s_u s_v}^s = 1$ or $c_{s_v s_u}^s = 1$.

Proof. It is straightforward to check that $c_{s_u s_1}^{s_v} = c_{s_u s_i}^{s_v} = 0$. Due to Lemma 6.1, we may assume that $s \in S_{max}$. Then by this lemma, $n_{s_u} = q-1 = n_{s_v}$. Therefore, $c_{s_u s}^{s_v} = c_{s_u s_v}^s$. The number $|H|c_{s_u s_v}^s$ is equal to the multiplicity, with which an element $w \in D_s$ enters the product

$$D_{s_u} D_{s_v} = HuH HvH = HU^\# HV^\# = HH(U^\# V^\#)$$

(see (40)). Thus, to prove statement (1), it suffices to verify that no two elements in UV belong to the same H -coset. For this aim, suppose that $u_1 v_1 h = u_2 v_2$ for

some $u_1, u_2 \in U$, $h \in H$, and $v_1, v_2 \in V$. Then the group U of unipotent upper triangular matrices contains the element

$$u_2^{-1}u_1 = v_2 h^{-1}v_1^{-1},$$

which is a low triangular matrix. It follows that $u_1 = u_2$, $v_1 = v_2$, $h = 1$, and we are done.

To prove statement (2), it suffices to verify that the complement to the set $D_{s_u}D_{s_v} \cup D_{s_v}D_{s_u}$ in G is equal to $D_{s_1} \cup D_{s_i} \cup D_{s_u} \cup D_{s_v}$. In view of equalities (40), this is equivalent to

$$(41) \quad G \setminus (HU^\#V^\# \cup HV^\#U^\#) = N \cup HU^\# \cup HV^\#.$$

To prove this relation, we observe that general elements of the sets $U^\#V^\#$ and $V^\#U^\#$ are, respectively,

$$\begin{pmatrix} 1+xy & x \\ y & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & x \\ y & 1+xy \end{pmatrix},$$

where x and y are nonzero elements in \mathbb{F}_q . Therefore, there are at least $q-1$ elements in $V^\#U^\#$, which do not belong to $U^\#V^\#$ (they correspond to nonzero elements xy). By the statement proved in the previous paragraph, it follows that the set $HU^\#V^\# \cup HV^\#U^\#$ is the disjoint union of $(q-1)^2$ distinct cosets of H contained in $HU^\#V^\#$ and at least $q-1$ distinct cosets of H contained in $HV^\#U^\#$. Since none of all these cosets is contained in $N \cup HU^\# \cup HV^\#$, we have

$$\begin{aligned} (q-1)^2q &= q(q+1)(q-1) - 2(q-1) - 2(q-1)^2 = \\ |G \setminus (N \cup HU^\# \cup HV^\#)| &\geq |HU^\#V^\# \cup HV^\#U^\#| \geq \\ (q-1)^2(q-1) + (q-1)(q-1) &= (q-1)^2q, \end{aligned}$$

which proves formula (41). \blacksquare

Let us verify that the coherent configuration \mathcal{X}_α with $\alpha = H$, is 1-regular. Indeed, in this case the first statement of Theorem 1.4 follows from Theorem 2.5 for $m = 2$, whereas the second statement is obvious.

To prove the 1-regularity of \mathcal{X}_α , it suffices to check that every point $\beta \in \alpha s_{max}$ is regular. However, if t is a basis relation of \mathcal{X}_α , then t is contained in a basis relation s of \mathcal{X} . If $s \in \{s_1, s_i\}$, then by Lemma 6.1, we have

$$|\beta t| \leq |\beta s| = n_s = 1.$$

Thus, by the same lemma, we may assume that $s \in S_{max}$ and hence t belongs to the set T_α defined in Lemma 3.2. By this lemma, all we need is to verify the connectedness of the graph s_α .

Let us prove the connectedness of s_α . Suppose that the pair $(\gamma, \delta) \in \alpha u \times \alpha v$, belongs to the basis relation s . Then, obviously, $c_{s_u s}^{s_v} \neq 0$. Thus, by statement (1) of Lemma 6.2 and the definition of s_α , the points γ and δ are adjacent in s_α . From statement (2) of Lemma 6.2, it follows that any other vertex $\beta \in \alpha s$ with $s \in S_{max}$, has at least one neighbor in the set $\alpha s_u \cup \alpha s_v$, i.e.,

$$\beta s_\alpha \cap (\alpha s_u \cup \alpha s_v) \neq \emptyset.$$

Thus, s_α is connected, and we are done.

7. PROOF OF THEOREM 1.5

We make use of the well-known Weisfeiler-Leman algorithm described in detail in [21, Section B]. The input of it is a set \mathcal{S} of binary relations on a set Ω , and the output is the smallest coherent configuration

$$\text{WL}(\mathcal{S}) = (\Omega, S)$$

such that $\mathcal{S} \subset S^\cup$. The running time of the algorithm is polynomial in the cardinalities of \mathcal{S} and Ω . The proof of the following statement is based on the Weisfeiler-Leman algorithm and can be found in [17, Theorem 3.5].

Theorem 7.1. *Let \mathcal{X} and \mathcal{X}' be coherent configurations on n points. Then given an algebraic isomorphism $\varphi : \mathcal{X} \rightarrow \mathcal{X}'$ all the elements of the set $\text{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ can be listed in time $(bn)^{O(b)}$ where $b = b(\mathcal{X})$.*

To solve the recognition problem, at first, we recognize the colored graphs D of Cartan schemes of G with respect to (B, N) -pair of rank at least 2. In this case, from the corollary of the main theorem in [16], it follows that B is the normalizer $N_G(P)$ of a group $P \in \text{Syl}_p(G)$ such that

$$(42) \quad H \cap P = 1, \quad H \leq N_G(P), \quad |N_G(P)| = |H| |P|,$$

where p is the characteristic of the ground field. By [19], one can also see that apart for a finite number of exceptional groups, $N = N_G(H)$ for every group from \mathcal{L} . Thus, the correctness of the following algorithm follows from Theorems 1.3 and 7.1.

In what follows, we denote by Ω the vertex set of the graph $D \in \mathcal{G}_n$, by \mathcal{S} the set of its color classes, and by $\mathcal{S}_{\alpha, \beta}$ the union of \mathcal{S} and the set of two one-element relations $\{(\alpha, \alpha)\}$ and $\{(\beta, \beta)\}$.

Recognizing Cartan schemes (the rank of (B, N) is at least 2)

Step 1. Find the coherent configuration $\mathcal{X} = \text{WL}(\mathcal{S})$.

Step 2. If there are no distinct points α, β such that the coherent configuration $\mathcal{X}_{\alpha, \beta} = \text{WL}(\mathcal{S}_{\alpha, \beta})$ is complete, then $b(\mathcal{X}) > 2$ and $D \notin \mathcal{K}_n$.

Step 3. Find all the elements of the group $G = \text{Iso}(\mathcal{X}, \mathcal{X}, \text{id})$ by the algorithm of Theorem 7.1. If G is not simple, then $D \notin \mathcal{K}_n$.

Step 4. Analyzing the number $|G|$, check that $G \in \mathcal{L}$. If not, then $D \notin \mathcal{K}_n$; otherwise set p to be the characteristic of the ground field associated with G .

Step 5. Fix a point stabilizer H of G and find $P \in \text{Syl}_p(G)$, for which relations (42) hold. If there is no such P , then $D \notin \mathcal{K}_n$.

Step 6. Now $D \in \mathcal{K}_n$ and \mathcal{X} is the Cartan scheme of G with respect to (B, N) , where $B = N_G(P)$ and $N = N_G(H)$. \blacksquare

Let us estimate the running time of the algorithm. At Steps 1 and 2, we apply the Weisfeiler-Leman algorithm $n(n-1)+1$ times. Thus, the complexity of these steps is at most $n^{O(1)}$. At Step 3, the time is polynomially bounded by Theorem 7.1 and the fact that a group is simple if and only if no nontrivial conjugacy class of it generates a proper subgroup (given the elements of G the conjugacy classes of it can be found efficiently). Step 4 requires polynomially many of arithmetic operations involving the number $|G|$ written in unary system. Here, we use the fact based on CFSG that except for known cases, any finite simple group is uniquely determined by its order (see Theorem 5.1 and Lemma 2.5 in [13]). Since Steps 5

and 6 can obviously be implemented in polynomial time for the group G given by the multiplication table, we conclude that the running time of the algorithm is at most $n^{O(1)}$.

The first four steps of the algorithm remain the same as before if we do not assume that the rank of (B, N) is at least 2. But in this case, one can find a 2-transitive representation of the group G ; here, a complete classification of all 2-transitive groups is useful (see, e.g., [10]). This enables to find the group B and N .

To solve the isomorphism problem, let $D \in \mathcal{K}_n$ and $D' \in \mathcal{G}_n$. Denote by \mathcal{S} and \mathcal{S}' the sets of color classes of D and D' , respectively. Without loss of generality, we may assume that there is a color preserving bijection $\psi : \mathcal{S} \rightarrow \mathcal{S}'$. Then one can apply the canonical version of the Weisfeiler-Leman algorithm presented in [21, Section M], where, in fact, the following statement was proved.

Theorem 7.2. *Let \mathcal{S} and \mathcal{S}' be m -sets of binary relations on an n -element set. Then given a bijection $\psi : \mathcal{S} \rightarrow \mathcal{S}'$ one can check in time $mn^{O(1)}$ whether or not there exists an algebraic isomorphism $\varphi : \text{WL}(\mathcal{S}) \rightarrow \text{WL}(\mathcal{S}')$ such that $\varphi|_{\mathcal{S}} = \psi$. Moreover, if φ does exist, then it can be found within the same time.* ■

Clearly, the original graphs D and D' are not isomorphic if there is no algebraic isomorphism φ from Theorem 7.2. Assuming the existence of φ , we can find the set

$$\text{Iso}(D, D') = \text{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$$

in time $(bn)^{O(b)}$ by Theorem 7.1, where $\mathcal{X} = \text{WL}(\mathcal{S})$, $\mathcal{X}' = \text{WL}(\mathcal{S}')$, and $b = b(\mathcal{X})$. Since $b \leq 2$ (Theorem 1.3), we are done.

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